THE ONSET OF SHOCK WAVES IN NONEQUILIBRIUM FLOWS

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A. N. KRAIKO
(Moscow)
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A large number of papers had been published up to the present which dealt with the analysis of shock wave formation in one-dimentional nonstationary flows with discontinuities resulting from the motion of a piston starting with zero velocity, and in a two-dimensional stationary (supersonic) flow with the shock wave generated by a concave wall with zero angle leading edge. The case of nonstationary plane waves considered in the early papers by Earnshaw [1], Hugoniot [2] and Rayleigh [3], and later by Pillow [4], found its way into numerous textbooks on gas dynamics (see, e. g., [5 and 6]). Jeffrey [7 and 8] had investigated a flow with cylindrical and spherical waves under homogeneous initial conditions of gas, and also of flows with a power law of density distribution inside cylindrical, or spherical cavities. Mayer [9] had analyzed the supersonic flow at the intake of a plane axisymmetric channel with a zero angle leading edge. Similar results were obtained by Johannesen [10] in the case of a plane flow. It was established that when either the piston acceleration, or the boundary curvature are finite and different from zero, the primary characteristic which bounds the perturbed flow area is intersected at a finite distance by the characteristics of the like set.

The position of the first intersection point on the primary characteristic is determined by the gas parameters, and either by the piston initial acceleration, or the curvature of the leading edge contour. All of the above-mentioned authors had limited their investigations to the equilibrium case, and usually considered a perfect gas of constant specific heat.

Papers by Bürger [11] and Rarity [12] were devoted to the determination of the first intersection point along the primary characteristic of equilibrium flows. Bürger had considered a one-dimensional nonstationary flow in the presence of plane waves. The gas thermodynamic properties, and the number of nonequilibrium processes were arbitrary. Although Rarity had confined himself to the case of a special "α-gas" with one nonequilibrium process, he had nevertheless considered, in addition to the problem solved by Bürger, a stationary plane supersonic flow. These investigations had shown that the presence of nonequilibrium processes results in the disappearance of the point of intersection of characteristics with the primary characteristic. With small, but finite relaxation times this point recedes into infinity, and with a still higher rate of these processes vanishes altogether.

This paper which presents in a generalized form the results of [11] and [12] deals with one-dimensional nonstationary flows with plane, cylindrical and spherical waves, and also with plain and axisymmetric stationary supersonic flows. The thermodynamic properties of the medium, and the number of nonequilibrium processes are assumed to be arbitrary, as was the case in [11]. External and internal problems are considered (we shall call a problem "internal" when dealing with a stationary flow in a channel, or with a nonstationary flow of gas in a bounded cavity).

The intersection points may generally occur not only along the primary characteristic, but also within the area of influence which leads to the formation of "internal" shock waves. The case in which "internal" shock waves are either absent, or so situated that they do not destroy the flow in the neighborhood of the first intersection point along the primary characteristic, which point is the origin of an "external" shock wave, belongs to a particular class of the laws of piston motion (or of the boundary form). The main distinctive feature of this case is that the location of the shock wave initial point may be determined from local considerations, without having to resort to a complete solution which even for equilibrium flows is, incidentally, known in the case of a plane flow only. Although the investigation of the position of the point of a shock wave origin in a nonequilibrium flow (as well as in the case of an equilibrium flow other than plane one) is possible in this particular case only, the obtained results are undoubtedly of interest, as they permit to establish, in particular, the trend of the influence of nonequilibrium on the formation of shock waves.

Problems here considered are examples of problems in which the failure of a continuous solution of a hyperbolic system of quasi-linear equations in two independent variables takes place. The failure of such solutions had recently been the subject of intensive investigations (see, e.g., [13] and the review by Lax [14]). In connection with the problem here considered, we should note between results of a general character the approach developed by Jeffrey [7 and 8], and by Jeffrey and Taniuty [15].

1. Let t denote time, r the distance of the axis, or the center of symmetry from a plane, and v the gas velocity (the velocity vector is directed along the r-axis). We shall consider a flow generated by a plane, cylindrical or spherical piston starting from rest with zero initial velocity subject to the law $r = R(t)$. If point O is the initial position of the piston, then with subscripts O denoting parameters at that point we obtain $r_0 = R(t_0)$ and $R_0' \equiv (dR/dt)_0 = 0$. We assume that at the beginning of motion the gas which fills space $r > r_0$, or cavity $r < r_0$ is at rest in a state of complete thermodynamic equilibrium, and that the distribution of its density ρ is uniform, i.e. is indedendent of r.

Let the thermodynamic state of the medium (its specific enthalpy h in particular) be determined by pressure p, density, and by n nonequilibrium parameters $q_1, \ldots, q_n$ which may represent the mass portions of the medium constituents, energies of internal degrees of freedom, etc. We shall denote a set of such parameters by q, and write expressions of the type of $f(q_1, \ldots, q_n)$ in the form $f(q)$.

A one-dimensional nonstationary nonequilibrium flow of an inviscid and non-heat-conducting gas is defined by Eqs.

$$\frac{\partial (r^\nu \rho)}{\partial t} + \frac{\partial (r^\nu \rho v)}{\partial r} = 0, \qquad \rho \frac{\partial v}{\partial t} + \rho v \frac{\partial v}{\partial r} + \frac{\partial p}{\partial r} = 0$$

$$\rho \frac{\partial h}{\partial t} + \rho v \frac{\partial h}{\partial r} - \frac{\partial p}{\partial t} - v \frac{\partial p}{\partial r} = 0, \qquad \tau_k \frac{\partial q_k}{\partial t} + \tau_k v \frac{\partial q_k}{\partial r} - \omega_k = 0 \quad (k = 1, \ldots, n) \tag{1.1}$$

Here $\nu = 0$, 1 or 2 for plane, cylindrical and spherical cases, respectively, and τ_k is a constant which has the meaning of relaxation time of parameters q_k. Functions

$$h = h(p, \rho, q), \quad \omega_k = \omega_k(p, \rho, q) \quad (k = 1, \ldots, n) \tag{1.2}$$

are assumed to be known. If $\tau_k = \infty$, then q_k is "frozen", i.e. remains at a constant value. If $\tau_k = 0$, then q_k is in equilibrium.

The equilibrium value of q_k is a function of p, ρ and $q_{j\neq k}$ in accordance with the final relationship

$$\omega_k(p, \rho, q) = 0 \qquad (1.3)$$

It is convenient to assume all parameters to be dimensionless. Let l_*, u_* and ρ_* be constants having respectively the dimensions of length, velocity and density. As constants we shall select, for example, the initial coordinate of the piston (with $\nu \neq 0$ and $r_0 \neq 0$), the velocity of sound, and the unperturbed gas density. If $\nu = 0$, or $r_0 = 0$, then product $u_* \tau_*$ where τ_* is the dimensionless relaxation time of one of the nonequilibrium processes, may be taken as the dimension of l_*. The reduction to a dimensionless form is accomplished by relating spatial variables to l_*, time to l_*/u_*, velocities to u_*, density to ρ_*, pressure to $\rho_* u_*^2$, and enthalpy to u_*^2. In the reduction of parameters q_k to a dimensionless form account is to be taken of their dimensions, and constants τ_k become dimensionless.

The system of Eqs. (1.1) is hyperbolic, and has in addition to particle trajectories $dr/dt = v$ two sets of real characteristics along which

$$\frac{dr}{dt} = v \pm a \qquad \left(a^2 = \frac{\rho h_\rho}{1 - \rho h_p}, \qquad \zeta_p = \left(\frac{\partial \zeta}{\partial p}\right)_{\rho,\, q}, \qquad \zeta_\rho = \left(\frac{\partial \zeta}{\partial \rho}\right)_{p,\, q}\right) \quad (1.4)$$

Here the upper (lower) sign corresponds to the characteristics of the first (second) set, while a is the "frozen" velocity of sound.

2. The grid of the first and second sets of characteristics cover the entire plane of variables rt. We shall define the characteristic variables ξ^1 and ξ^2 by Eq.

$$\frac{\partial \xi^i}{\partial t} + (v \pm a)\frac{\partial \xi^i}{\partial r} = 0 \qquad (i = 1,\ 2) \qquad (2.1)$$

where the upper sign corresponds to $i = 1$, and the lower to $i = 2$. It follows from (2.1) and (1.4) that the value of ξ^i remains constant along every characteristic of the ith set. We pass from variables rt over the semicharacteristic variables $r\xi^i$. In terms of these new variables system (1.1) and the supplementary relationship defining t are written in the form

$$p_r \pm \rho a v_r + \frac{a^2}{v \pm a}\left(\frac{\nu \rho v}{r} - \sum_{k=1}^{n} \frac{h_k \omega_k}{\tau_k h_\rho}\right) = 0$$

$$\rho a^2 v_\xi \mp a p_\xi + \rho a v (a \pm v) v_r t_\xi - (v \pm a) v p_r t_\xi + (a \pm v)^2 p_r t_\xi = 0 \qquad (2.2)$$

$$a^3 \rho_\xi - a p_\xi + a^2 v (a \pm v) \rho_r t_\xi - v(a \pm v) p_r t_\xi + a^2 (a \pm v) t_\xi \sum_{k=1}^{n} \frac{h_k \omega_k}{\tau_k h_\rho} = 0$$

$$a \tau_k q_{k\xi} + \tau_k v (a \pm v) q_{kr} t_\xi = (a \pm v) t_\xi \omega_k \qquad (k = 1, \ldots, n)$$

$$t_r = (v \pm a)^{-1}$$

$$\left(\varphi_r = \left(\frac{\partial \varphi}{\partial r}\right)_{\xi^i}, \qquad \varphi_\xi = \left(\frac{\partial \varphi}{\partial \xi^i}\right)_r, \qquad h_k = \left(\frac{\partial h}{\partial q_k}\right)_{p,\, \rho,\, q_{j\neq k}}\right)$$

The upper signs in (2.2) yield a system which defines the flow in terms of variables $r\xi^1$, and the lower ones in terms of variables $r\xi^2$.

Let the piston trajectory be represented in the rt-plane by the solid shaded curve emanating from point O, as shown on Fig. 1, where the right-hand (left-hand) branch defines the piston motion for the case of an external (internal) problem. Characteristics ξ^1 = const, and ξ^2 = const of the external and internal problems respectively are shown by solid lines, and in each case the characteristic of the corresponding set emanating from point O (we shall call this characteristic "primary", and assume that along it $\xi^1 = 0$) separates the perturbed flow area (above it) from the area at rest. The primary characteristic, as well as the characteristics in the area at rest, is a staight line along which the derivatives of v_ξ, p_ξ, ρ_ξ, a_ξ and t_ξ become discontinuous. At the same time, as is seen from (2.2), $q_{l\xi}$, and all derivatives with respect to r remain continuous. We shall denote by subscript ∞ the derivatives which remain continuous, and parameters of the gas when $\xi^1 = 0$. From assumptions made about the gas at rest, Eqs. (1.3) which at complete thermodynamic equilibrium are valid for all k, and about the equations for q_k in (2.2), we find

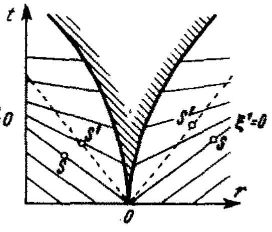

Fig. 1

$$v_\infty = \omega_{k\infty} = v_{r\infty} = p_{r\infty} = \rho_{r\infty} = a_{r\infty} = q_{l\,r\infty} = q_{l\,\xi\infty} = 0 \qquad (2.3)$$

Intersection of characteristics of the ith set occur at those points of the rt-plane at which the Jacobian of transformation of the independent variables, i.e.

$$\frac{D(r,\,t)}{D(r,\,\xi^i)} \equiv t_\xi = 0 \qquad (2.4)$$

We shall call such points singular, and denote parameters at these by subscript s. Let us assume that on each of the characteristics of the ith set lying above the primary characteristic we have ξ^1 equal to $(t - t_0)$ at points of intersection of these with the piston trajectory, and at points of intersection with the straight line $r = r_0$ for those lying below the primary characteristic. With this assumption the area of perturbed flow will be defined by inequality $\xi^1 > 0$, and the area at rest by $\xi^1 < 0$, and here $t_\xi \equiv 1$. Although characteristic $\xi^1 = 0$ is the discontinuity line of t_ξ, nevertheless, by virtue of the law of piston motion ($R_0' = 0$) here considered, and because of the selected value of ξ^1, we have at point O, as well as above this characteristic

$$t_{\xi 0} = 1 \qquad (2.5)$$

Here, and throughout the following text the values of all parameters at $\xi^1 = 0$ are understood to be taken at $\xi^1 = +0$, i.e. as upper limit values.

At the piston surface the velocity of gas is $R'(t)$. Therefore, in accordance with the selection of ξ^1

$$v_{\xi 0} = R_0'' \qquad (2.6)$$

System (2.2) together with the derived initial conditions and equalities (2.3) completely define the values of derivatives t_ξ, p_ξ, v_ξ etc. on the upper side of the primary characteristic. This is so because, first of all, from the second and third of Eqs. (2.2) written for $\xi^1 = +0$ and equality $a = a(p, \rho, q)$ with (2.3) taken into account, we have

$$\rho_\xi = \frac{1}{a_\infty^2}\,p_\xi, \qquad v_\xi = \pm\frac{1}{\rho_\infty a_\infty}\,p_\xi, \qquad a_\xi = \frac{a_{\rho\infty} + a_\infty^2 a_{p\infty}}{a_\infty^2}\,p_\xi \qquad (2.7)$$

Here and in the following the upper sign relates to the external problem, and the lower to the internal one.

Similarly, differentiating in the first of Eqs. (2.2) with respect to ξ^1, and the second with respect to r, utilizing (2.3) and (2.7), and eliminating $v_{\xi r}$ from obtained equations, we find that for $\xi^1 = +0$ p_ξ is defined by the differential Eq.

$$p_{\xi r} = \left(-\frac{\nu}{2r} \pm \lambda_\infty\right) p_\xi \qquad \left(\lambda = \sum_{k=1}^{n} \frac{h_k(\omega_{k\rho} + a^2\omega_{kp})}{2\tau_k a h_\rho}\right) \tag{2.8}$$

The integration of (2.8) with initial condition (2.6) rewritten for p_ξ with the aid of (2.7) yields

$$p_\xi = \pm \rho_\infty a_\infty R_0'' \left(\frac{r_0}{r}\right)^{\nu/2} \exp\left[\pm \lambda_\infty (r - r_0)\right] \quad \text{for } \xi^i = +0 \tag{2.9}$$

In the plane case ($\nu = 0$) the internal and external problems coincide with an accuracy which depends on the selection of the direction of the r-axis, while it may be always assumed that $r_0 = 0$. Taking the piston initial coordinate as the characteristic dimension, we obtain $r_0 = 1$ for cylindrically and spherically symmetric flows ($\nu = 1$ and 2) and $r_0 > 0$. It is interesting to note that in the case of piston expansion from either the axis, or the center of symmetry ($r_0 = 0$), and with a finite R_0'' and $\nu \neq 0$, we have all along the primary characteristic $p_\xi \equiv 0$.

The equation defining t_ξ for $\xi^1 = +0$ is obtained by the differentiation of the last of Eqs.(2.2) with respect to ξ^1 followed by a transformation with (2.3) and (2.7) taken into account, and is of the form

$$t_{\xi r} = \mp \frac{\alpha_\infty}{\rho_\infty a_\infty} p_\xi \qquad \left(\alpha = \frac{a + \rho a_\rho + \rho a^2 a_p}{a^3}\right)$$

Substituting into this p_ξ from (2.9) and integrating with respect to r with initial condition (2.5), we find that

$$t_\xi = 1 - R_0'' \alpha_\infty \Phi(\nu, r, r_0, \lambda_\infty) \quad \text{при } \xi^i = +0 \tag{2.10}$$

$$\Phi(\nu, r, r_0, \lambda_\infty) = \int_{r_0}^{r} \left(\frac{r_0}{z}\right)^{\nu/2} \exp\left[\pm \lambda_\infty (z - r_0)\right] dz$$

If all nonequilibrium processes are frozen ($\tau_k = \infty$, with $k = 1, \ldots, n$), then $\lambda_\infty = 0$, and

$$\Phi(\nu, r, r_0, 0) = \begin{cases} r - r_0 & \text{for } \nu = 0 \\ 2\left[(r r_0)^{1/2} - r_0\right] & \text{for } \nu = 1 \\ r_0 \ln(r / r_0) & \text{for } \nu = 2 \end{cases} \tag{2.11}$$

The same formulas hold for a complete equilibrium flow ($n = 0$), however, the equilibrium values of a_∞ and α_∞ which we shall denote by $a_{e\infty}$ and $\alpha_{e\infty}$ differ from a_∞ and α_∞, as by virtue of (1.3) all q_k's under equilibrium conditions must be considered as functions of p and ρ. It is known that $a_\infty > a_{e\infty}$, therefore, the primary characteristic under equilibrium condition (straight dotted line on Fig. 1) lies above the similar characteristic of a nonequilibrium flow.

In conditions of incomplete equilibrium, when only the last $n - r$ parameters ($\tau_{r+1} = \ldots = \tau_n = 0$) change in an equilibrium manner, the primary characteristic occupies an intermediate position.

Cases of complete and incomplete equilibrium may be considered to be the limits of the nonequilibrium case when $\tau_j \to 0$ for $j = r + 1, \ldots, n$, where $r \geq 0$. It should be borne in mind that because of the stipulation of stability of the equilibrium state [16]

$$-\infty \leqslant \lambda_\infty \leqslant 0$$

where $\lambda_\infty = 0$ in the "frozen" case considered above, and $\lambda_\infty = -\infty$ in the case of a complete, or partial equilibrium. Because $\Phi(\nu, r, r_0, -\infty) = 0$, hence, from (2.9) and

(2.10) we obtain $p_\xi \equiv 0$, $t_\xi \equiv 1$ when $\xi^1 = +0$, which means that in accordance with was stated previously the flow is unperturbed not only below, but also in a certain neighborhood above the primary characteristic, when $\lambda_\infty = -\infty$.

When $r = r_0$, $\Phi = 0$ by definition, and for finite rates of nonequilibrium processes, i.e. when $\lambda_\infty \neq -\infty$, with $r \neq r_0$ the following inequalities hold:

$$\Phi \gtrless 0, \qquad \left(\frac{\partial \Phi}{\partial r}\right)_{\nu, r_0, \lambda_\infty} > 0, \qquad \left(\frac{\partial \Phi}{\partial \lambda_\infty}\right)_{\nu, r, r_0} \gtrless 0$$

Thus, in an external problem Φ is positive and increases with increasing r, while in an internal one Φ is negative and its absolute value increases with decreasing r. The decreases of λ_∞ from 0 to $-\infty$ in both cases results in a decrease of $|\Phi|$. If $r_0 = 0$, and $\nu = 1$, or 2, then $\Phi \equiv 0$.

The integral in the expression of Φ may be obtained not only for the limit value of λ_∞, but also for any arbitrary of its values. To sum up, we have

$$\Phi(\nu, r, r_0, \lambda_\infty) =$$

$$= \begin{cases} \pm \lambda_\infty^{-1} (\exp [\pm \lambda_\infty (r - r_0)] - 1) & \text{for } \nu = 0 \\ (\mp \pi r_0 / \lambda_\infty)^{1/2} [\mathrm{H}(\sqrt{\mp \lambda_\infty r}) - \mathrm{H}(\sqrt{\mp \lambda_\infty r_0})] \exp(\mp \lambda_\infty r_0) & \text{for } \nu = 1 \\ r_0 [\mathrm{Ei}(\pm \lambda_\infty r) - \mathrm{Ei}(\pm \lambda_\infty r_0)] \exp(\mp \lambda_\infty r_0) & \text{for } \nu = 2 \end{cases} \tag{2.12}$$

$$\left(\mathrm{H}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2)\, dt, \quad \mathrm{Ei}(x) = \int_{-\infty}^x \frac{\exp t}{t}\, dt\right)$$

A comparison of (2.11) and (2.12) shows that contrary to the case of a "frozen" flow in which Φ increases infinitely with increasing r, the limit value of Φ for $\lambda_\infty < 0$ and $r \to \infty$ is finite. In the case of an external problem this property is of great importance for the elucidation of the question of existence of a singular point belonging to the primary characteristic.

The condition defining coordinate r_s for $\xi^1 = 0$ is written in accordance with (2.4) and (2.10) in the form

$$1 - R_0'' \alpha_\infty \Phi(\nu, r_s, r_0, \lambda_\infty) = 0 \tag{2.13}$$

As α_∞ is always positive (for a perfect gas with adiabatic exponent $\varkappa$ it is equal $(\varkappa + 1)/2a_\infty^2$), the intersection of characteristics occurs, as in the case of equilibrium flows, only when the piston moves towards the gas, i.e. when $R_0'' \gtrless 0$.

Denoting by r_s^0 the value of r_s corresponding to the "frozen" case ($\lambda_\infty = 0$), we obtain from (2.11) and (2.13)

$$r_s^0 = \begin{cases} r_0 + (R_0'' \alpha_\infty)^{-1} & \text{for} \quad \nu = 0 \\ r_0^{-1} [r_0 + {}^1/_2 (R_0'' \alpha_\infty)^{-1}]^2 & \text{for} \quad \nu = 1 \\ r_0 \exp (r_0 R_0'' \alpha_\infty)^{-1} & \text{for} \quad \nu = 2 \end{cases} \tag{2.14}$$

For a perfect gas Eq. (2.14) coincides with results published in [4 to 8].

Increasing $|\lambda_\infty|$ from zero to infinity leads by virtue of the properties of function Φ to a recession of point s, i.e. to an increase of $|r_s - r_0|$. In the case of the external problem there exists a finite value of λ_∞ determined by condition (2.13) with $r_s = \infty$ for which point s recedes into infinity. There are no singular points on the primary characteristic for large values of $|\lambda_\infty|$. A similar situation occurs in the case of an inner problem with cylindrical waves, for which the limiting value of λ_∞ is found from (2.13) with $r_s = 0$. On the other hand, $\Phi(2,\ 0,\ r_0,\ \lambda_\infty) = -\infty$, and consequently, in the case of an inner problem with spherical symmetry, Eq. (2.13) defines a certain finite $r_s > 0$, for any arbitrary $\lambda_\infty > -\infty$.

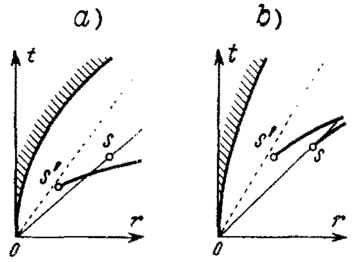

Fig. 2 a, b

The absence of a singular point on the nonequilibrium primary characteristic does not preclude the existence of a shock wave originating within the influence area, even when in the case of a "frozen" flow ($\lambda_\infty = 0$) there are no "internal" shock waves. Thus, for example, when $\lambda_\infty = -\infty$, i.e. in the limit equilibrium case, there exists an "internal" singular point (s' on Fig. 1) lying on the primary equilibrium characteristic. In accordance with what was said before it follows that the coordinate $r_{s'}$ is also found from (2.14), but with $\alpha_{e\infty}$ substituted for α_∞. Usually $\alpha_{e\infty} > \alpha_\infty$, therefore $| r_{s'} - r_0 | < | r_s^0 - r_0 |$. As a rule an opposite inequality exists for $t_{s'}$ and t_s^0. A similar pattern will also be observed in the case of sufficiently large, but finite absolute values of λ_∞, although the determination of point s' of the shock wave onset requires further consideration. The derived relationships are, of course, not applicable when an "internal" shock wave intersects the primary nonequilibrium characteristic, as shown on Fig. 2a where the shock wave is represented by the solid line emanating from point s'. The converse situation is also possible (for example, on account of selection of the piston motion law) in which case there are two shock waves (Fig. 2 b).

The perturbation of flow in the neighborhood of the primary nonequilibrium characteristic may be defined for $\xi^1 = +0$ by the magnitude of the derivative $(\partial p / \partial t)_r \equiv p_\xi / t_\xi$. In accordance with (2.9), (2.10) and (2.12) the absolute value of this derivative with $\lambda_\infty < 0$ first decreases with increased distance from point O the faster, the greater $|\lambda_\infty|$ is, and then sharply increases in the neighborhood of s, or of the axis of symmerty, and becomes infinite. It follows from this that with $|\lambda_\infty(r_s - r_0)| \gg 1$ there exists between points O and s an area of almost unperturbed gas, and that the intensity of the shock wave generated in the neighborhood of s is low.

In concluding the analysis of the nonstationary case, we may note that, if its solution does exist, equality (2.13) uniquely defines r_s or λ_∞ by virtue of the monotonousness of Φ with respect to r and λ_∞, when the remaining parameters are known. In particular, we have for $\nu = 0$

$$r_s = r_0 \pm \frac{1}{\lambda_\infty} \ln\left(1 \pm \frac{\lambda_\infty}{R_0'' \alpha_\infty}\right)$$

which coincides to within notations used with the results obtained by Bürger [11].

3. Let x and y be orthogonal coordinates which in the axisymmetric case lie in the meridian plane, with the x-axis coinciding with the axis of flow. We denote the projections of gas velocity on the x- and y-axes by u and v respectively. We shall consider a plane, or axisymmetric flow of homogeneous supersonic stream in a state of thermodynamic equilibrium past a body the contour of which is generated by curve $y = Y(x)$. Let the flow of gas be from left to right, and the tangent to the generatrix at the leading edge be parallel to the velocity of the free stream (Fig. 3, where O is the leading edge of the body, with the upper and lower branches of the solid shaded line represent the generatrices for the external and internal problems respectively).

Therefore $y_0 = Y(x_0)$ and $Y_0' \equiv (dY / dx)_0 = 0$. The thermodynamic properties of gas are assumed to be the same as in the nonstationary case.

The system of equations of a two-dimensional nonequilibrium stationary flow has, except along streamlines on which $dy / dx = v / u$ two sets of real characteristics at supersonic velocities, i.e. when $w^2 \equiv u^2 + v^2 > a^2$ along which we have

$$\frac{dy}{dx} = \frac{uv \pm a \sqrt{w^2 - a^2}}{u^2 - a^2} \tag{3.1}$$

Here the upper (lower) sign defines the direction of characteristics of the first (second) set.

As in the nonstationary case we introduce characteristic variables ξ^1 and ξ^2 defined by Eq.

$$\frac{\partial \xi^i}{\partial x} + \frac{uv \pm a \sqrt{w^2 - a^2}}{u^2 - a^2} \frac{\partial \xi^i}{\partial y} = 0 \qquad (i = 1, 2) \tag{3.2}$$

where the plus sign is taken for $i = 1$ and minus for $i = 2$. It is obvious from (3.1) and (3.2) that on each characteristic of an ith set ξ^i = const.

The flow analyzed here is expressed in terms of semicharacteristic variables $y\xi^i$ by the system of equations

$$uv_y - vu_y \pm \frac{\sqrt{w^2 - a^2}}{\rho a} p_y + \frac{N}{y^\nu \rho}\left(\frac{\nu v}{y} - \sum_{k=1}^{n} \frac{h_k \omega_k}{\tau_k \rho h_\rho}\right) = 0$$

$$uv_\xi - vu_\xi \mp \frac{\sqrt{w^2 - a^2}}{\rho a} p_\xi + \frac{N x_\xi}{y^\nu \rho^2 u^2} (\rho u v v_y - \rho v^2 u_y + u p_y) = 0$$

$$\rho u u_\xi + \rho v v_\xi + p_\xi + \frac{v N x_\xi}{y^\nu \rho u^2} (\rho u u_y + \rho v v_y + p_y) = 0$$

$$w^2 + 2h = \text{const} \tag{3.3}$$

$$u \tau_k q_{k\xi} + \frac{N x_\xi}{y^\nu \rho u} (v \tau_k q_{ky} - \omega_k) = 0 \qquad (k = 1, \ldots, n)$$

$$x_y = \frac{u^2 - a^2}{uv \pm a \sqrt{w^2 - a^2}} \qquad \left(\varphi_y = \left(\frac{\partial \varphi}{\partial y}\right)_{\xi^i}, \ \varphi_\xi = \left(\frac{\partial \varphi}{\partial \xi^i}\right)_y, \ N = \frac{y^\nu \rho a (av \pm u \sqrt{w^2 - a^2})}{uv \pm a \sqrt{w^2 - a^2}}\right)$$

In (3.3) $\nu = 0$, or 1, respectively, in a plane, or axisymmetric case, with the upper signs yielding a system representing the flow in terms of variables $y\xi^1$, and the lower – in terms of variables $y\xi^2$.

The enthalpy h, velocity of sound a, functions ω_k and the remaining thermodynamic parameters are defined by the previously used relationships.

Characteristics ξ^1 = const, and ξ^2 = const of the external and internal problems respectively are shown on Fig. 3 by solid lines. In each of these cases we shall assume that along the primary characteristic emanating from the point O and bounding the area of unperturbed flow we put $\xi^i = 0$. As in the nonstationary case we shall assign subscript ∞ to derivatives which remain continuous along the primary characteristic, as well as to gas parameters for $\xi^i = 0$

$$v_\infty = \omega_{k\infty} = u_{y\infty} = v_{y\infty} = p_{y\infty} = \rho_{y\infty} = a_{y\infty} = q_{ky\infty} = q_{k\xi\infty} = 0 \tag{3.4}$$

We select on each characteristic of the ith set ξ^i equal to $(x - x_0)$ at points of intersection of these either with the generatrix of the body, or with the straight line $y = y_0$, depending on whether the characteristics lie to the right, or left of the primary characteristic. The area of the perturbed flow will then be defined by inequality $\xi^i > 0$, while at the point O we shall have for $\xi^1 = +0$

$$x_{\xi 0} = 1, \qquad v_{\xi 0} = u_\infty Y_0'' \tag{3.5}$$

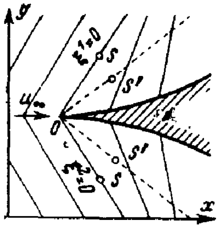

Fig. 3

The points of intersection of characteristics, which we shall as previously call singular, are determined by equality

$$x_\xi = 0$$

The condition for the determination of coordinates of a singular point belonging to the primary characteristic is found in the same manner as in the nonstationary case.

When $\xi^1 = +0$ we obtain from (3.3) and (3 4)

$$\rho_\xi = \frac{1}{a_\infty^2} p_\xi, \qquad u_\xi = -\frac{1}{\rho_\infty u_\infty} p_\xi, \qquad v_\xi = \pm \frac{\sqrt{M_\infty^2 - 1}}{\rho_\infty u_\infty} p_\xi$$

$$a_\xi = \frac{a_{\rho\infty} + a_\infty^2 a_{p\infty}}{a_\infty^2} p_\xi, \quad p_{\xi y} = \left(-\frac{\nu}{2y} \pm \lambda_\infty\right) p_\xi, \quad x_{\xi y} = \mp \alpha_\infty \frac{\sqrt{M_\infty^2 - 1}}{\rho_\infty u_\infty^2} p_\xi \tag{3.6}$$

$$\left(\lambda = \frac{u}{\sqrt{u^2 - a^2}} \sum_{k=1}^{n} \frac{h_k(\omega_{k\rho} + a^2 \omega_{kp})}{2\tau_k a h_\rho}, \quad \alpha = \frac{u^4 (a + \rho a_\rho + \rho a^2 a_p)}{a^3 (u^2 - a^2)}\right)$$

where $M = w/a$ is the Mach number. The upper (lower) sign relates here and in the following to the external (internal) problem.

Integrating the last two of Eqs. (3.6) with initial conditions (3.5) taken into account, we find that for $\xi^1 = +0$

$$p_\xi = \pm Y_0'' \frac{\rho_\infty u_\infty^2}{\sqrt{M_\infty^2 - 1}} \left(\frac{y_0}{y}\right)^{\nu/2} \exp\left[\pm \lambda_\infty (y - y_0)\right]$$

$$x_\xi = 1 - Y_0'' \alpha_\infty \Phi(\nu, y, y_0, \lambda_\infty)$$

where Φ coincides with the similar function in (2.10).

Formulas for p_ξ and x_ξ differ from (2.9) and (2.10) only with respect to notations, and unimportant positive multiplication factors appearing in the expressions of p_ξ, α and λ. Hence all deductions of the preceding section related to $\nu = 0$ and 1 are valid in this case also (the equilibrium characteristic appearing in our deductions is also shown on Fig. 3 by a dotted line). If we substitute x for t, and y for r, and Y for R, then formulas subsequent to (2.10) are also valid.

In concluding we may note that for a perfect gas

$$\alpha_\infty = \frac{(\varkappa + 1) M_\infty^4}{2(M_\infty^2 - 1)}, \quad \lambda_\infty = 0$$

and (2.14) leads to expressions for y_s obtained in [9 and 10).

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